# PROBLEMS ON INFINITE PRODUCTS

BESLIKAS ATHANASIOS

Abstract. In this paper we will investigate some infinite products using the fundamental identity of complex analysis, $z = |z|\, e^{i\theta}$. We will present our formulas in the form of problems, in order to help the reader.

## Problem 1

**Problem 0.1.** *Prove that:*

$$\prod_{n=1}^{\infty} \frac{1 + itan(\frac{1}{n^2\pi})}{sec(\frac{1}{n^2\pi})} = e^{\frac{i\pi}{6}} \tag{0.1}$$

*Proof.* It is true that $\frac{z}{|z|} = e^{i\theta}$ , where $\theta = Arg(z) = arctan\left(\frac{Im(z)}{Re(z)}\right)$. Hence we have:

$$\prod_{n=1}^{\infty} \frac{z_n}{|z_n|} = \prod_{n=1}^{\infty} e^{i\theta} \Rightarrow \prod_{n=1}^{\infty} \frac{z_n}{|z_n|} = exp\left(i \sum_{n=1}^{\infty} arctan\left(\frac{Im(z_n)}{Re(z_n)}\right)\right) \tag{0.2}$$

Assuming that:

$$z_n = 1 + itan\left(\frac{1}{n^2\pi}\right)$$

we have:

$$\prod_{n=1}^{\infty} \frac{1 + itan\left(\frac{1}{n^2\pi}\right)}{\sqrt{1 + tan^2\left(\frac{1}{n^2\pi}\right)}} = exp\left(\sum_{n=1}^{\infty} arctan\left(tan\left(\frac{1}{n^2\pi}\right)\right)\right) \tag{0.3}$$

$$\prod_{n=1}^{\infty} \frac{1 + itan\left(\frac{1}{n^2\pi}\right)}{\sqrt{1 + tan^2\left(\frac{1}{n^2\pi}\right)}} = e^{\sum_{n=1}^{\infty} \frac{1}{n^2\pi}} \tag{0.4}$$

Hence:

$$\prod_{n=1}^{\infty} \frac{1 + itan(\frac{1}{n^2\pi})}{sec(\frac{1}{n^2\pi})} = e^{\frac{i\pi}{6}} \tag{0.5}$$

□

What we saw in this problem was a very fundamental proof to an easy problem. Let us investigate a more difficult problem using as a basis the previous problem.

**Problem 0.2.** *Prove the following inequality:*

$$\prod_{n=1}^{\infty} \left( \frac{sec\left(\frac{1}{p_n^2}\right)}{1 + itan\left(\frac{1}{p_n^2}\right)} \right)^{\frac{1}{i}} \leq \frac{\pi^2}{15},\ p_n\ prime. \tag{0.6}$$

*Proof.* We assume the following product:

$$\prod_{n=1}^{\infty} \left( \frac{1 + itan\left(\frac{1}{p_n^2}\right)}{sec\left(\frac{1}{p_n^2}\right)} \right) = exp\left( i \sum_{n=1}^{\infty} \frac{1}{p_n^2} \right) \tag{0.7}$$

from the previous solution. Then we have:

$$\prod_{n=1}^{\infty} \left( \frac{1 + itan\left(\frac{1}{p_n^2}\right)}{sec\left(\frac{1}{p_n^2}\right)} \right)^{\frac{1}{i}} = exp\left( \sum_{n=1}^{\infty} \frac{1}{p_n^2} \right) \tag{0.8}$$

Using the following inequality we have:

$$\sum_{k=1}^{n} a_n \leq \prod_{k=1}^{n} (1 + a_n) \leq exp\left( \sum_{n=1}^{\infty} \frac{1}{p_n^2} \right) \Rightarrow \tag{0.9}$$

from equation (0.8) we have:

$$exp\left( \sum_{n=1}^{\infty} \frac{1}{p_n^2} \right) \geq \prod_{n=1}^{\infty} \left( 1 + \frac{1}{p_n^2} \right) \tag{0.10}$$

which is a known product from Ramanujan[1] . Hence, we have that:

(0.11)

$$\prod_{n=1}^{\infty}\left(\frac{1+itan\left(\frac{1}{p_n^2}\right)}{sec\left(\frac{1}{p_n^2}\right)}\right)^{\frac{1}{i}} \geq \prod_{n=1}^{\infty}\left(1+\frac{1}{p_n^2}\right)=\frac{15}{\pi^2} \Rightarrow \prod_{n=1}^{\infty}\left(\frac{sec\left(\frac{1}{p_n^2}\right)}{1+itan\left(\frac{1}{p_n^2}\right)}\right)^{\frac{1}{i}} \leq \frac{\pi^2}{15}.$$

□

## References

1. Wolfram Math World, *mathworld.wolfram.com*

Department of Mathematics, Aristotle University of Thessaloniki, 541 24 Thessaloniki

*E-mail address*: mpeslikas@math.auth.gr